

COMPARING THE AUTOMORPHISM GROUP OF THE MEASURE ALGEBRA WITH SOME GROUPS RELATED TO THE INFINITE PERMUTATION GROUP OF THE NATURAL NUMBERS

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ABSTRACT. We prove, by a straight construction, that the automorphism group of the measure algebra and the subgroup of the measure preserving ones cannot be isomorphic to the trivial automorphisms of $\mathcal{P}(\mathbb{N})/\text{fin}$.

1. INTRODUCTION

This work concerns the problem of comparing $\text{Aut}\mathcal{M}$, the automorphism group of the measure algebra, with some groups related to the infinite symmetric group of ω , the natural numbers. More precisely:

Problem 1. It is possible to embed T_ω^* , the trivial automorphisms of $\mathcal{P}(\mathbb{N})/\text{fin}$, inside $\text{Aut}\mathcal{M}$, or even inside $\text{MPAut}\mathcal{M}$, the subgroup of measure preserving ones?

and the natural converse:

Problem 2. It is possible to embed $\text{Aut}\mathcal{M}$, or at least $\text{MPAut}\mathcal{M}$, inside T_ω^* ?

Observe that S_ω , the infinite permutation group without any quotient with some ideals (fin in the above case), can be embedded inside $\text{Aut}\mathcal{M}$, even if, hardly ever, inside $\text{MPAut}\mathcal{M}$. This drives to the hard problem of understanding how the structure of a group changes in correspondence of the ideal of its quotient. This seems to be an automorphisms version of a problem widely explored, for example by I. Farah [2].

The problem of embedding measure algebra in $\mathcal{P}(\mathbb{N})/\text{fin}$ it has been widely investigated [see [1], for example] revealing the deep differences between them, even if the result depends on the choice of the axioms. A question about their automorphism groups naturally arises. It seems that their difference drastically increases.

In [8] it is showed that T_ω^* is not simple. This is because the quotient T_ω^*/S_ω^* is isomorphic to \mathbb{Z} , where S_ω^* is the subgroup of the permutation

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group of ω modulo a finite number of exchanges, known as “very trivial automorphisms”. On the contrary S_ω^* is simple [7].

On the other side $\text{Aut}\mathcal{M}$ is simple by a general result on homogeneous Dedekind complete BA [see [3] 381T]. The same can be said for $\text{MPAut}\mathcal{M}$ by a result on homogeneous algebra totally finite [see Ibidem 382I(b)], therefore T_ω^* fails to be isomorphic to both of them.

In the following we provide with a concrete map between the underlying algebras whenever an isomorphism between two of these groups, $\text{Aut}\mathcal{M}$, $\text{MPAut}\mathcal{M}$, T_ω^* , S_ω^* , occurs.

This map turns to be injective and chain preserving, which drives to a contradiction when we admit the existence of an isomorphism between $\text{Aut}\mathcal{M}$, $\text{MPAut}\mathcal{M}$ and T_ω^* , S_ω^* , by the different structure of $\mathcal{P}(\mathbb{N})/\text{fin}$ and \mathcal{M} .

The construction of this map is performed using a technique for reconstructing an algebra from a group of its automorphisms [[6] and [5]].

Actually we believe in a stronger result, namely that both of the cited problems have a negative answer, which roughly speaking means that permutations of infinite numbers of naturals and reals have a different structure.

2. HOW TO BUILD A MAP BETWEEN THE UNDERLYING ALGEBRAS WHENEVER AN ISOMORPHISM BETWEEN THEIR AUTOMORPHISM GROUPS IS GIVEN.

For a complete and general survey on the technique for reconstructing an algebra from a group of their automorphisms, refer to [6].

Assume \mathcal{B} is a boolean algebra[BA] and Σ the supremum operation.

Definition 2.1. $a, b, c \in \mathcal{B}$, $f, g, h \in \text{Aut}(\mathcal{B})$

$$\begin{aligned} \text{var}(f) &=_{\text{def}} \Sigma\{a \in \mathcal{B} \mid f(a) \cdot a = 0\}. \\ \text{fix}(f) &=_{\text{def}} \Sigma\{a \in \mathcal{B} \mid (\forall b \leq a)(f(b) = b)\}. \end{aligned}$$

Note that $-\text{var}(f) = \text{fix}(f)$.

$$\begin{aligned} \text{Sp}_G(a) &=_{\text{def}} \{g \in G \subseteq \text{Aut}(\mathcal{B}) \mid \text{var}(g) \leq a\}. \\ \text{Sp}_G^*(a) &=_{\text{def}} \{g \in G \subseteq \text{Aut}(\mathcal{B}) \mid \text{var}(g) = a\}. \\ Z(f) &=_{\text{def}} \{g \mid gf = fg\}. \\ f^h &=_{\text{def}} hfh^{-1}. \\ [f, g] &= fgf^{-1}g^{-1}. \\ [h, f, g] &=_{\text{def}} [[h, f], g]. \end{aligned}$$

The following Lemma shows that even if $\mathcal{P}(\mathbb{N})/\text{fin}$ is not a complete BA, the supremum $\text{var}(f)$ does exist for all f in T_ω^* , which turns to be the only one we shall need.

Lemma 2.2. *For all f in T_ω^* the set $\text{var}(f)$ belongs to $\mathcal{P}(\mathbb{N})/\text{fin}$.*

Proof. Let $\mathcal{A} = \Sigma\{a \in \mathcal{A} \mid f(a) \cap a = \emptyset\}$. Consider the following set $F = \{n \in \mathbb{N} \mid f(n) = n\}$. We show that $\text{var}(f) = \mathbb{N} \setminus F$.

Consider $E = \mathbb{N} \setminus F$, by Katetov theorem (for example [4]) E can be decomposed into three disjoint sets E_0, E_1, E_2 in such a way $f[E_0] \subseteq E_1$, $f[E_1] \subseteq E_2$ and $f[E_2] \subseteq E_0$.

First observe that for all $B \subseteq E$ B cannot be almost disjoint from all elements of \mathcal{A} .

Indeed, there must exists i such that $B \cap E_i = A$ is a set of infinite size. By construction $f[A] \cap A = \emptyset$.

On the other side if $B \cap E = \emptyset$ B must be contained into F therefore B is disjoint from all elements of \mathcal{A} . This in particular implies $\text{fix}(f) = F$. \square

Now consider the following two formulas:

Definition 2.3.

$$\begin{aligned} \varphi_1(f, f') &\equiv \forall g ([g, f] \neq Id) \\ &\quad \rightarrow (\exists f_1 f_2 \in Z(f')) ([g, f_1, f_2] \neq Id) \wedge ([g, f_1, f_2], f'] = Id)) \\ \varphi_{\leq}(f, g) &\equiv V(f) \subseteq V(g) \end{aligned}$$

where

$$\begin{aligned} D_1(f) &= \{f' \mid \varphi_1(f, f') \text{ holds} \} \\ V(f) &=_{\text{def}} Z(\{(f')^4 \mid f' \in D_1(f)\}) \end{aligned}$$

In order to define an injective chain preserving map between measure algebra and $\mathcal{P}(\mathbb{N})/\text{fin}$ we show the following result, whose proof, except for a slight modification, can be seen in [5]. For reader's convenience we report it in the appendix. Observe that along these proofs we will use only one type of occurrence of the supremum operator whose existence is guaranteed in both algebras, by the completeness of measure algebra and the previous lemma.

Theorem 2.4. $\varphi_{\leq}(f, g)$ holds in T_{ω}^* or in S_{ω}^* (respectively in $\text{Aut}\mathcal{M}$ or in $\text{MPAut}\mathcal{M}$) if and only if it holds $\text{var}(f) \leq \text{var}(g)$ in $\mathcal{P}(\mathbb{N})/\text{fin}$ (respectively in \mathcal{M}).

Definition 2.5. $\varphi_{=}(f, g) \equiv \varphi_{\leq}(f, g) \wedge \varphi_{\leq}(g, f)$.

Corollary 2.6.

$$\begin{aligned} \varphi_{=}(f, g) \text{ holds in } T_{\omega}^*, S_{\omega}^*, \text{Aut}\mathcal{M}, \text{MPAut}\mathcal{M} \\ \Leftrightarrow \text{var}(f) = \text{var}(g). \end{aligned}$$

Now we are ready for the announced result.

Theorem 2.7. The groups $\text{Aut}\mathcal{M}$, $\text{MPAut}\mathcal{M}$ cannot be isomorphic either T_{ω}^* and S_{ω}^* .

Proof. It is sufficient to perform the proof in the case of S_ω^* and $\text{Aut}\mathcal{M}$, since in other cases the proof runs exactly in the same manner. Let $a \in \mathcal{P}(\mathbb{N})/\text{fin}$ consider $Sp_G^*(a)$, observe that it is certainly not empty. Let Φ be an isomorphism between the two groups S_ω^* and $\text{Aut}\mathcal{M}$. For a fixed $a \in \mathcal{P}(\mathbb{N})/\text{fin}$ the element $\text{var}(f')$ such that $f' \in \Phi[Sp_G^*(a)]$ is uniquely determined. By Corollary 2.6 for all $f, g \in Sp_G^*(a)$ $\varphi_=(f, g)$ holds in S_ω^* , since $\varphi_=-$ is expressed in pure group language, $\varphi_=(\Phi(f), \Phi(g))$ holds as well. Since, by Corollary 2.6, $\text{var}(f')$ does not depend on the choice of $f' \in \Phi[Sp_G^*(a)]$, the following definition makes sense:

Let $a \in \mathcal{P}(\mathbb{N})/\text{fin}$, pick an $f' \in \Phi[Sp_G^*(a)]$ and define $\Theta(a) = \text{var}(f')$.

By Theorem 2.4 this map is injective and easily preserves the length of chains, but in $\mathcal{P}(\mathbb{N})/\text{fin}$ there are ω_1 -chains, the same cannot be said for \mathcal{M} , a contradiction. \square

Remark 2.8. The same result can be generalized using Rubin's terminology [see [5]] in the following manner:

Theorem 2.9. *Assume \mathcal{M} is a complete atomless BA which does not contain ω_1 -chains and G is a locally moving subgroup of $\text{Aut}(\mathcal{M})$ then T_ω^* and S_ω^* cannot be isomorphic to G .*

3. APPENDIX: PROOF OF THEOREM 2.4

Henceforth the group G could be any of $\text{Aut}\mathcal{M}$, $\text{MPAut}\mathcal{M}$, T_ω^* , S_ω^* .

Lemma 3.1. *Let $k_0 \dots k_n \in \mathbb{Z}$, $f \in \text{Aut}(B)$ and $a \in B$, where B is a BA. Assume that $f^{k_0}(a), \dots, f^{k_n}(a)$ are mutually disjoint, then for any $h_1, \dots, h^n \in Z(f)$ and $0 \neq b \leq a$ the following holds*

$$\sum_{i=0}^n f^{k_i}((b)) \not\leq \sum_{i=1}^n h_i(a)$$

Proof. By induction on n .

Base case. Using $f^{k_0}(b) + f^{k_1}(b) \leq h_1(a)$, we get $b + f^{-k_0+k_1}(b) \leq f^{-k_0}h_1(a)$, analogously we deduce that $b \leq f^{-k_1}h_1(a)$. Therefore $f^{-k_0}h_1(a) \cdot f^{-k_1}h_1(a) \neq 0$, hence, by the fact that $h_1 \in Z(f)$, $h_1(f^{-k_0}(a) \cdot f^{-k_1}(a)) \neq 0$, and finally $f^{-k_0}(a) \cdot f^{-k_1}(a) \neq 0$, a contradiction.

Inductive case. If $h_{n+1}(a) \cdot \sum_{i=1}^n f^{k_i}((b)) = 0$, by inductive hypothesis we are done. Otherwise let j such that $c = h_{n+1}(a) \cdot f^{k_j}(b) \neq 0$. Now we build a sequence of b_i corresponding to f^{k_i} , $i \neq j$ (relabel the sequence in order to make easier the construction), in such a way $b_{i+1} \leq b_i \leq b$ and $h_{n+1}(a) \cdot f^{k_i}(b) = 0$. We can assume $j = 0$ and $b' = f^{-k_j}(c)$. Observe that the theorem holds for all $l < n + 1$ in particular for 1, hence $h_{n+1}(a) \not\leq f^{k_0}(b') + f^{k_j}(b') = f^{k_0}(b') + c$. By construction $h_{n+1}(a) \geq c$, therefore $h_{n+1}(a) \not\leq f^{k_0}(b')$, thanks to this we can define $b_0 = f^{-k_0}(f^{k_0}(b') \setminus h_{n+1}(a))$. Suppose we have already built the first i objects, again $h_{n+1}(a) \not\leq f^{k_{i+1}}(b_i) + f^{k_j}(b_i)$. Observe that $h_{n+1}(a) \geq f^{k_j}(b_0) \geq f^{k_j}(b_i)$, indeed $h_{n+1}(a) \geq c \geq$

$c \setminus f^{k_j - k_0} h_{n+1}(a) = f^{k_j}(f^{-k_j}(c) \setminus f^{-k_0} h_{n+1}(a)) = f^{k_j}(b' \setminus f^{-k_0} h_{n+1}(a))$,
hence $h_{n+1}(a) \not\leq f^{k_{i+1}}(b_i)$, which makes sense to the following definition:

$$b_{i+1} = f^{-k_{i+1}}(f^{k_{i+1}}(b_i) \setminus h_{i+1}(a)),$$

satisfying the requested properties and

$$h_{n+1}(a) \cdot \sum_{i \neq j}^n f^{k_i}((b_i)) = 0.$$

Since the collection of functions without f^{k_j} and h_{n+1} satisfies inductive hypothesis we have:

$$\sum_{i \neq j}^{n+1} f^{k_i}((b_i)) \not\leq \sum_{i=1}^n h_i(a),$$

hence

$$\sum_{i \neq j}^{n+1} f^{k_i}((b_i)) \not\leq \sum_{i=1}^{n+1} h_i(a),$$

and finally

$$\sum_{i=0}^{n+1} f^{k_i}((b_i)) \not\leq \sum_{i=1}^{n+1} h_i(a).$$

□

Lemma 3.2. (a) $\text{var}(f) \cdot \text{var}(f') = 0$ implies $\varphi_1(f, f')$;
(b) $\text{var}(f) \cdot \text{var}(f'^4) \neq 0$ implies $\neg \varphi_1(f, f')$.

Proof. (a). g does not commute with f this yields $\text{var}(f) \cdot \text{var}(g) \neq 0$. Pick an $a \leq \text{var}(f) \cdot \text{var}(g)$ such that $g(a) \cdot a = 0$ and let $f_1 \in Sp_G(a)$. Since $\text{var}(f_1) \leq \text{var}(f)$ f_1 commutes with f' . Consider $[g, f_1] = f_1^g f_1^{-1} = g_1$; observe that $g(\text{var}(f_1)) \cdot \text{var}(f_1) = 0$ this, by standard arguments, implies $\text{var}(g_1) = \text{var}(f_1) + g(\text{var}(f_1))$. $g_1(\text{var}(f_1))$ turns to be equal to $\text{var}(f_1)$, indeed $g_1(\text{var}(f_1)) = g f_1 g^{-1} f_1^{-1}(\text{var}(f_1)) = g f_1 g^{-1}(\text{var}(f_1))$, since $g^{-1}(\text{var}(f_1)) \cdot \text{var}(f_1) = 0$ we get $g g^{-1}(\text{var}(f_1))$. Since $\text{var}(g_1) = \text{var}(f_1) + g(\text{var}(f_1))$, we can pick a $b \leq \text{var}(g_1) \cdot \text{var}(f_1)$ in such a way $g_1(b) \cdot b = 0$, again let $f_2 \in Sp_G(b)$. Since $\text{var}(f_2) \leq \text{var}(f)$ f_2 commutes with f' . Define $[g_1, f_2] = f_2^{g_1} f_2^{-1} = g_2$; as before from $g_1(\text{var}(f_2)) \cdot \text{var}(f_2) = 0$ we get $\text{var}(g_2) = \text{var}(f_2) + g_1(\text{var}(f_2))$, moreover $\text{var}(f_2) \leq \text{var}(f_1)$ and $g_1(\text{var}(f_1)) = \text{var}(f_1)$ therefore $\text{var}(g_2) \leq \text{var}(f_1)$ and commutes with f' , and we are done.

(b). Let $a = \text{var}(f) \cdot \text{var}(f'^4)$, since $a \leq \text{var}(f'^4) \leq \text{var}(f'^3) \leq \text{var}(f'^2) \leq \text{var}(f')a \leq \prod_{i=1}^n \text{var}(f'^i)$ therefore there exists by Lemma 3.3 (a) $b \leq a$ such that $b \cdot \sum_{i=1}^n f^i(b) = 0$. In particular for any $j < i \leq 4$, since $b \cdot f^{i-j}(b) = 0$, $f^j(b) \cdot f^i(b) = 0$. By Lemma 3.3 (d) there exists $g \in Sp_G(b)$ which does not commute with f . We are now to show that for any $f_1, f_2 \in Z(f')$

$g_2 = [g, f_1, f_2] = \text{Identity}$ or $[g_2, f'] \neq \text{Identity}$. Assume the former is not, we show the latter.

$$\begin{aligned}
g_2 &= [g(g^{-1})^{f_1}, f_2] \\
&= g(g^{-1})^{f_1} f_2 (g(g^{-1})^{f_1})^{-1} f_2^{-1} \\
&= g(g^{-1})^{f_1} ((g(g^{-1})^{f_1})^{-1})^{f_2} \\
&= g(g^{-1})^{f_1} ((g^{-1})^{f_1})^{-1} g^{-1} f_2 \\
&= g(g^{-1})^{f_1} ((f_1 g f_1^{-1})^{-1} g^{-1})^{f_2} \\
&= g(g^{-1})^{f_1} (f_1 g f_1^{-1} g^{-1})^{f_2} \\
&= g(g^{-1})^{f_1} (g^{f_1} g^{-1})^{f_2} \\
&= g(g^{-1})^{f_1} (f_2 f_1 g f_1^{-1} g^{-1} f_2^{-1}) \\
&= g(g^{-1})^{f_1} (f_2 f_1 g f_1^{-1} f_2^{-1} f_2 g^{-1} f_2^{-1}) \\
&= g(g^{-1})^{f_1} g^{f_1 f_2} (g^{-1})^{f_2}
\end{aligned}$$

which in turns implies

$$\begin{aligned}
\text{var}(g_2) &\leq \text{var}(g) + \text{var}((g^{-1})^{f_1}) + \text{var}(g^{f_1 f_2}) + \text{var}((g^{-1})^{f_2}) \\
&= b + f_1(b) + f_1 f_2(b) + f_2(b).
\end{aligned}$$

The above inequality shows that $\text{var}(g_2)$ intersect in b at least one h chosen among $\text{Identity}, f_1, f_2, f_1 f_2$. Let $c = h(b) \cdot \text{var}(g_2)$, moreover $h \in Z(f')$ in any case (this is because $f_1, f_2 \in Z(f')$), which in turns implies that $\{f'^i(h(b)) \mid i : 1 \dots 4\}$ are mutually disjoint.

By contradiction $[g_2, f'] = \text{Identity}$ then $g_2^{f'^i} = g_2$ for any i . Using $c \leq \text{var}(g_2)$ we deduce:

$$\text{var}(g_2) = \sum_1^n (\text{var}(g_2))^{f'^i} = \sum_1^n f'^i((\text{var}(g_2))) \geq \sum_1^n f'^i((c)).$$

Observe that $h(b) \geq c$ therefore $\{f'^i(c) \mid i : 1 \dots 4\}$ are mutually disjoint, as well. Moreover $f_1 h^{-1}, f_2 h^{-1}, f_1 f_2 h^{-1} \in Z(f')$ and Lemma 3.1 applies, showing that:

$$\begin{aligned}
\sum_1^n f'^i((c)) &\not\leq h^{-1}(h(b)) + f_1 h^{-1}(b) + f_2 h^{-1}(b) + f_1 f_2 h^{-1}(b) \\
&\geq \text{var}(g_2) \\
&\geq \sum_1^n f'^i((c)),
\end{aligned}$$

a contradiction. □

Lemma 3.3. (a) $0 \neq a \leq \prod_1^n \text{var}(g_i)$ implies there exists $b \leq a$ such that $b \cdot \sum_1^n g_i(b) = 0$;

- (b) $a \neq 0$ and $n \in \mathbb{N}^+$ implies there exists $h \in Sp_G(a)$ such that $h^n \neq \text{Identity}$;
- (c) $0 \neq a \leq \text{var}(f) \cdot \text{var}(g)$ implies there exists $h \in Sp_G(a)$ such that f^h does not commute with g ;
- (d) Let $g \in G$ and $\text{var}(g) \geq a \neq 0$. Then there is $k \in Sp_G(a)$ such that k does not commute with g .

Proof. (a) is proved by an induction on n . (d) plainly follows from (c).

(b). By induction on n we shall show that there exists an $h \in Sp_G(a)$ and $0 \neq b \leq a$ such that $b, h(b), \dots, h^n(b)$ are mutually disjoint.

This is certainly true for $n = 1$ since either the algebras are homogeneous and h is not the Identity. Assume it is true for n , if h^{n+1} restricted to b is different from the Identity, we can choose $0 \neq c \leq b$ such that $h^{n+1}(c) \cdot c = 0$. Indeed, $h^{n+1}(c) \cdot h^i(c)$ is equal to $h(h^n(c) \cdot h^{i-1}(c))$ which is 0 by the induction hypothesis. Otherwise h^{n+1} restricted to b is the Identity map, in this case choose $k \in Sp_G(b)$ and $0 \neq c \leq b$ such that $k(c) \cdot c = 0$. Let $g = kh$ obviously $h(c) \cdot b$ is equal 0 by inductive hypothesis, therefore $h(c)$ is outside the variation of k this means the in $h(c)$ k is the Identity, hence $g^i(c) = \underbrace{khkh \dots kh}_{i-1}h(c)$, for the same reasons $h^2(c) \cdot b$ is equal 0, following

in the same manner we get $g^i(c) = h^i(c)$. We are left to prove that $g^{n+1}(c)$ is disjoint from $g^i(c)$ for all $i : 0 \dots n$. Observe that, by the assumption that h^{n+1} restricted to b is the Identity map, $g^{n+1}(c) = kh^{n+1}(c) = k(c)$, which is disjoint from c by construction. Finally, since k has its variation inside b $g^{n+1}(c) \leq b$, by inductive hypothesis $g^{n+1}(c)$ is disjoint from $g^i(c)$ for all $i : 0 \dots n$.

(c). If f does not commute with g we are already done since we can choose Identity map as h . Otherwise choose $b_1 \leq a$ in such a way either $f(b_1) \cdot b_1$ and $g(b_1) \cdot b_1$ are 0. Therefore $(f(b_1) + g(b_1)) \cdot b_1 = 0$.

If $b_1 \cdot \text{var}(fg) \neq 0$ consider $b \leq b_1$ such that $fg(b) \cdot b = 0$ (Case 1). Otherwise define $b = b_1$ (Case 2).

By (b) it is possible choosing $h \in Sp_G(b)$ in such a way $h^2 \neq \text{Identity}$. Let $c \leq b$ such that $c, h(c), h^2(c)$ are mutually disjoint. We shall show that g and f^h does not commute on $h(c)$. Observe that $gf^h(h(c)) = ghfh^{-1}h(c) = ghf(c)$, in both cases $b \cdot f(c) = 0$ since $c \leq b \leq b_1$, therefore restricted to $f(c)$ h is the Identity, hence $gf^h(h(c)) = gf(c)$.

On the other side $f^hg(h(c)) = hfh^{-1}g(h(c))$, since $h(c) \leq \text{var}(h) \leq b$ and $g(b) \cdot b = 0$ $gh(c) \cdot b = 0$, but h^{-1} has the same variation as h therefore on $h^{-1}gh(c)$ is the Identity. Hence $f^hg(h(c)) = hfg(h(c))$.

Case 1. $h(c) \leq b$, therefore $fg(h(c)) \cdot b = 0$, which in turns implies that h on $fg(h(c))$ is the Identity and $f^hg(h(c)) = fgh(c)$. But f and g commutes so $f^hg(h(c)) = gfh(c)$. $f^hg(h(c))$ is equal to $gf^h(h(c))$ this entails $gfh(c) = gf(c)$ and $c = h(c)$, a contradiction.

Case 2. fg is the Identity on b , since f and g commutes and both c and $h(c)$ are inside b , $f^h g(h(c)) = h^2(c)$ and $gf^h(h(c)) = c$ which should imply $c = h^2(c)$, again a contradiction. \square

Proof on Theorem 2.4. Assume $V(f) = Sp_G(\text{var}(f))$. In this case it is obvious to deduce that $\text{var}(f) \leq \text{var}(g)$ implies $\varphi_{\leq}(f, g)$. On the other side if $\text{var}(f) \leq \text{var}(g)$ does not hold there exists an $h \in G \cdot Sp_G(\text{var}(f) \setminus \text{var}(g))$ therefore $h \in Sp_G(\text{var}(f))$ and $h \notin Sp_G(\text{var}(g))$ which implies the negation of $\varphi_{\leq}(f, g)$.

We are left to prove $V(f) = Sp_G(\text{var}(f))$.

Observe that whenever $\text{var}(f)$ and $\text{var}(g)$ are disjoint they commute.

Consider the first inclusion $V(f) \supseteq Sp_G(\text{var}(f))$. Let $g \in Sp_G(\text{var}(f))$ and $f' \in D_1(f)$ we are to show that they commute. Since $f' \in D_1(f)$ $\varphi_1(f, f')$ holds. By Lemma 3.2 (b) $\text{var}(f)$ and $\text{var}(f'^4)$ are mutually disjoint, observe that $\text{var}(g) \subseteq \text{var}(f)$ hence $\text{var}(g)$ and $\text{var}(f'^4)$ are disjoint, as well, which implies the thesis.

Now we show $\neg V(f) \supseteq \neg Sp_G(\text{var}(f))$.

Consider $g \in \neg Sp_G(\text{var}(f))$ therefore $\text{var}(g) \setminus \text{var}(f) = b$ is not empty. By Lemma 3.3 (b) there exists an f' inside $Sp_G(b)$ such that $f'^4 \neq \text{Identity}$. $\text{var}(g)$ contains b and $\text{var}(f'^4) \subseteq \text{var}(f') \subseteq b$ therefore $\text{var}(g) \cdot \text{var}(f'^4) = b'$ is not empty. Lemma 3.3 (c) ensures the existence of a morphism h inside $Sp_G(b')$ such that $(f'^4)^h$ does not commute with g . Observe that $\text{var}((f'^4)^h) = h(\text{var}(f'^4))$ but the variation of h lies inside the variation of f' hence $\text{var}((f')^h) = \text{var}(f')$ which in turns implies $\text{var}((f')^h) \cdot \text{var}(f) = \emptyset$. Our claim is to show that $g \notin V(f)$ that means there exists a morphism inside $D_1(f)$ which does not commute with g ; $(f')^h$ is such an object. Indeed, by Lemma 3.2 (a) using $\text{var}((f')^h) \cdot \text{var}(f) = \emptyset$ we get $\varphi_1(f, (f')^h)$, hence $(f')^h \in D_1(f)$ but $((f')^h)^4$ does not commute with g . \square

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